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A relation between the temperature exponents of the eight-vertex and q-state Potts model

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Abstract. The mapping of the critical points in the q-state model for $q \leq 4$ onto the Baxter line in the eight-vertex model makes it possible, by a comparison of the exactly known critical exponent y_T^{8v} with the approximately known values for y_T^{P}, to conjecture the relation $(y_T^{8v}-2)(y_T^{\mathrm{P}}-3)=3$. This relation confirms weak universality.

1. Introduction

In this paper a relation between the critical exponents for the temperature operator of the eight-vertex model and the q-state Potts model (for $q \leq 4$) is proposed:

$$(y_T^{8v}-2)(y_T^{\mathrm{P}}-3)=3. \tag{1.1}$$

This relation can be rewritten, using the Baxter solution, as an explicit equation for y_T^{P}:

$$y_T^{\mathrm{P}}(q)=\tfrac{3}{2}[2+\pi/(\mu-\pi)] \qquad \text{for } q \leq 4 \text{ with } \cos\mu=\tfrac{1}{2}\sqrt{q} \tag{1.2}$$

(the specific heat exponent is related to y_T as $\alpha=2-2/y_T$).

The q-state Potts model, eight-vertex model and Ashkin–Teller model have in common (see § 2) that their critical lines can all be mapped on the $T>T_c$ domain of the F model (solved by Lieb (1967)). The temperature operators of the three models correspond to three different directions in which one can leave the F model. In other words, one can construct a parameter space in which the three models intersect each other at their critical line; along this line (the Baxter line) the three models reduce to the F model. The identification of the fields is summarised in table 1.

These equivalences were already implicitly known in the literature; no new mappings are needed to show them. A combined presentation however, as given in § 2, with the F model as central model cannot be found in the literature.

A comparison in § 3 of the critical exponent y_T^{P} of the q-state Potts model, as obtained by recent approximative calculations, with the exactly known y_T^{8v} (Baxter 1971) lead then to the conjecture that these two exponents satisfy a simple relation along the Baxter line (i.e. for $q \leq 4$).

Equation (1.1) is similar to relations between other critical exponents along the Baxter line that have been conjectured before (see § 3). Just like these other relations, equation (1.1) does not contain the parameter of the Baxter line explicitly. It confirms the concept of weak universality.

Table 1. Operators for the F model.

$$(\omega_1, \omega_2, \omega_3, \omega_4; \omega_5, \omega_6, \omega_7, \omega_8) = \begin{cases} (a\,e^{u}, a\,e^{u}, a\,e^{-u}, a\,e^{-u}; e^{s}, e^{-s}; d, d) & \text{A sublattice} \\ (a\,e^{-u}, a\,e^{-u}, a\,e^{u}, a\,e^{u}; e^{-s}, e^{s}, d, d) & \text{B sublattice} \end{cases}$$

Field	Interpretation at $u = s = d = 0$	Behaviour of the free energy
a	Temperature in F model	Infinite-order transition at $a = \frac{1}{2}$ (*Lieb* 1967)
s	Staggered electric field (staggered polarisation of F model)	$a > \frac{1}{2}$: $f \sim \lvert s\rvert^{2/y_s^F}$; $a < \frac{1}{2}$: first-order (Baxter 1973a)
d	Temperature in eight-vertex model	$a > \frac{1}{2}$: $f \sim \lvert d\rvert^{2/y_T^{8v}}$, $y_T^{8v} = 2\mu/\pi$ (Baxter 1971); $a < \frac{1}{2}$: regular, $\cos\mu = 1/2a^2 - 1$
u	Temperature in AT model	$a > \frac{1}{2}$: $f \sim \lvert u\rvert^{2/y_T^{AT}}$; $a < \frac{1}{2}$: regular
$s = (i/2)\tan(\mu/2)\epsilon_P$, $u = \frac{1}{2}\epsilon_P$	Temperature in Potts model	$a > \frac{1}{2}$: $f \sim \lvert\epsilon_P\rvert^{2/y_T^P}$; $a < \frac{1}{2}$: first-order (Baxter 1973b)

2. Relations between the eight-vertex, Ashkin–Teller and Potts models

In this section it is discussed how the eight-vertex model, the Ashkin–Teller model and the q-state Potts models are successively related to the F model. The definition and interpretation of the several fields are summarised in table 1. All mappings used in this section are already known, but appear rather scattered in the literature. This section serves as a review of these mappings in a presentation that has the F model as central model.

2.1. *The eight-vertex model*

The eight-vertex model has been introduced as a model for (anti)ferroelectrics. Draw arrows on the edges of a two-dimensional square lattice, with the restriction that an even number of arrows point into every vertex. Attach to each of the allowed vertex states (see figure 1) a Boltzmann weight ω_i. The symmetric eight-vertex model, solved by Baxter (1971), is obtained when $\omega_1 = \omega_2 = a$, $\omega_3 = \omega_4 = b$, $\omega_5 = \omega_6 = c$ and $\omega_7 = \omega_8 = d$.

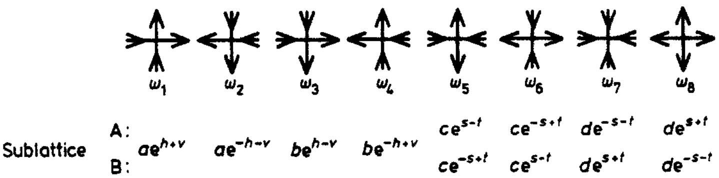

Fig. 1. The eight different vertex configurations and their Boltzmann weights.

For $a \gg b, c, d$ and for $b \gg a, c, d$ the arrows will order ferroelectrically, while for $c \gg a, b, d$ and $d \gg a, b, c$ an antiferroelectric groundstate is obtained. Notice that in each of these four limits the arrows are allowed to order in two groundstates (two coexisting phases). So in the eight-vertex model, two natural order parameters play a role: the polarisation and the staggered polarisation. Conjugate to these two order parameters are respectively the direct electric fields h and v (figure 1) and the staggered fields s and t (for the definition of a staggered field, divide the lattice in the usual way into two sublattices A and B). In the Ising translation of the eight-vertex model (Wu 1971, Kadanoff and Weger 1971) another order parameter emerges: the magnetisation (to which the magnetic field is conjugated). The electrical fields are translated there into the nearest-neighbour interactions.

Baxter (1971) gives the solution of the symmetrical eight-vertex model in the so-called principal domain:

$$a>0, \qquad b>0, \qquad d>0, \qquad c>a+b+d. \tag{2.1}$$

The free energy shows a singularity at the $c=a+b+d$ border only. The critical exponent varies continuously in this plane:

$$f \sim |T-T_c|^{2/y_T^{8v}} \tag{2.2a}$$

$$y_T^{8v} = 2\mu/\pi, \qquad \cos\mu = (ab-cd)/(ab+cd). \tag{2.2b}$$

The solution in the rest of the (a, b, c, d) space can now be constructed from the symmetry relations given by Fan and Wu (1970):

$$Z(a, b; c, d)$$

$$= Z(a, b; c, -d) \tag{2.3a}$$

$$= Z(b, a; c, d) \tag{2.3b}$$

$$= Z(c, d; a, b) \tag{2.3c}$$

$$= Z(\tfrac{1}{2}(a-b+c+d), \tfrac{1}{2}(-a+b+c+d); \tfrac{1}{2}(a+b+c-d), \tfrac{1}{2}(a+b-c+d)). \tag{2.3d}$$

In figure 2 the result in the $a=b$, $c=1$ plane is shown (the $c=1$ choice is no restriction).

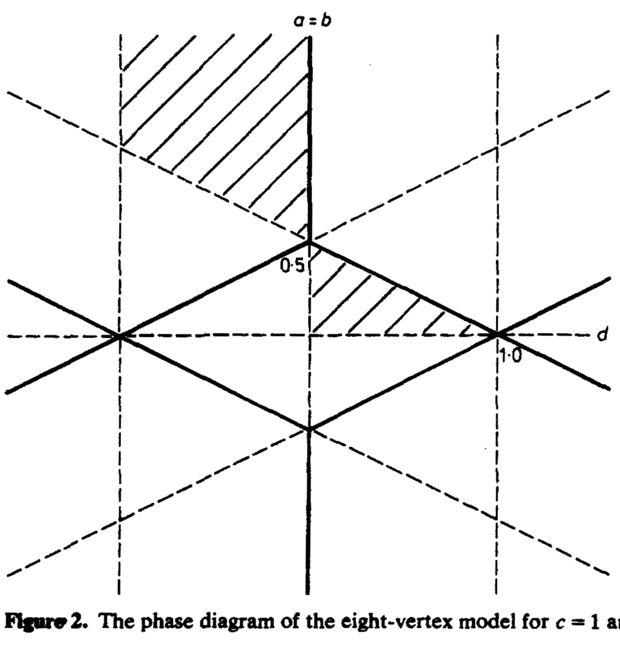

Figure 2. The phase diagram of the eight-vertex model for $c=1$ and $a=b$.

The full curves are critical curves corresponding to the Baxter line; the broken curves correspond to noncritical borders of the principal domain. The two shaded 'triangles' are mapped on each other by relation (2.3d).

For $d=0$ (the Ice condition) and $a=b$, the eight-vertex model reduces to the F model (already solved by Lieb (1967)). He found an infinite order transition at $a=\frac{1}{2}$ (i.e. $y_T^F=0$). From figure 2 we conclude that the Baxter line in the principal domain is mapped on the $T>T_c$ domain of the F model†.

Variation of the temperature in the F model corresponds to moving along the $a=b$ line in the critical plane $1=a+b+d$ of the eight-vertex model.

Moreover, by solving the Ice models Lieb has automatically solved the eight-vertex model at the critical plane and the noncritical borders in the principal domain. The whole $1=a+b+d$ plane is mapped by equation (2.3d) onto the $d=0$ plane. From the solution of the Ice models as discussed by Lieb and Wu (1972), we learn that a transition in the Ice models is found when lines corresponding to the boundary of the critical domain in the $1=a+b+d$ plane (i.e. at $a=0$, $b=0$ or $d=0$) are crossed. This transition is F-like (i.e. antiferroelectric; infinite order) when $\Delta=-1$ ($d=0$) at the border and KDP-like (i.e. ferroelectric; first order) when $\Delta=+1$ (a or $b=0$). This variable Δ (introduced by Lieb) is related to the μ variable (introduced by Baxter) as

$$\Delta=-\cos\mu. \tag{2.4}$$

We return now to the F model. Below T_c the staggered polarisation (the order parameter) shows a jump (Baxter 1973a). So for $a<\frac{1}{2}$ a first-order transition in the staggered field direction s is found. The F model shows long-range correlations, not only at T_c but also for all $T>T_c$. The staggered susceptibility is expected to diverge with a continuously varying exponent, i.e. the free energy is singular in the s direction as

$$f\sim|s|^{2/y_s^F}. \tag{2.5}$$

This exponent is only known exactly at $a=\frac{1}{2}\sqrt{2}$, where Baxter (1970) found the second derivative with respect to s to diverge logarithmically ($y_s^F=1$), and at $a=\frac{1}{2}$ where $y_s^F=1{\cdot}5$ (Baxter 1973a).

Finally, from the mapping of equation (2.3d) and the Baxter solution, it follows that along the F model for $T>T_c$

$$f\sim|d|^{2/y_T^{8v}} \tag{2.6a}$$

with

$$y_T^{8v}=2\mu/\pi, \qquad \cos\mu=1/2a^2-1 \tag{2.6b}$$

while for $T<T_c$ the free energy remains regular in the d direction (see table 1).

2.2. *The Ashkin–Teller model*

The Ashkin–Teller model has been introduced to describe a mixture of four components A, B, C and D (Ashkin and Teller 1943).

Associate (Fan 1972) to every vertex i of the two-dimensional square lattice two Ising spins (s_i, t_i). Then the four different states of a vertex can be identified as

† After completion of this work, the author received a preprint by Temperley and Ashley (1978) where this mapping of the Baxter line onto the F model, and the conclusion that the critical line in the Potts model is a Baxter line, is also established.

A = (+, +), B = (−, +), C = (+, −) and D = (−, −). The Hamiltonian is chosen as

$$H=\sum_{\langle ij\rangle} K\delta_{s_is_j}\delta_{t_it_j}+J_1\delta_{s_is_j}+J_2\delta_{t_it_j}. \tag{2.7}$$

This implies for two nearest-neighbour vertices, four different Boltzmann weights: $w_0=\exp(K+J_1+J_2)$ for AA, BB, CC and DD, $w_1=\exp(J_1)$ for AB and CD, $w_2=\exp(J_2)$ for AC and BD, $w_3=1$ for AD and BC.

Fan (1972) has derived a duality transformation for this model, which leaves the $w_0=w_1+w_2+w_3$ plane invariant. By a duality transformation on the t_i spins only, however, the Ashkin–Teller (AT) model is mapped on another Ising model which is equivalent to a staggered eight-vertex model. The equations obtained by Wegner (1972) read (after an additional application of equation (2.3*c*) and a normalisation of *c*):

$$\begin{aligned} a^{\mathrm{A}}&=b^{\mathrm{B}}=(w_2+w_3)/(w_0+w_1)\\ b^{\mathrm{A}}&=a^{\mathrm{B}}=(w_0-w_1)/(w_0+w_1)\\ c&=1\\ d&=(w_2-w_3)/(w_0+w_1). \end{aligned} \tag{2.8}$$

A and B refer to the two sublattices (see above). One can introduce a staggered field u such that $a^{\mathrm{A}}=b^{\mathrm{B}}=a\exp(u)$ and $a^{\mathrm{B}}=b^{\mathrm{A}}=a\exp(-u)$, and

$$2u=\ln[(w_2+w_3)/(w_0-w_1)] \tag{2.9}$$

When $u=0$ the AT model reduces to the symmetrical eight-vertex model with $a=b$. From equations (2.8) we read that $u=0$ corresponds to $w_0=w_1+w_2+w_3$, i.e. the plane in the AT model that is invariant under duality (Fan 1972).

The intersection of this plane with the $w_1=w_2=w_3$ line (i.e. $J_1=J_2=0$, the four-state Potts model) is mapped on the transition point $a=\frac{1}{2}$ in the F model. We conclude from figure 2 that the dual plane of the AT model must contain three Baxter lines that meet each other at this point. They are situated (Wu and Lin 1974) along $w_1=w_2$ for $w_3<\frac{1}{3}w_0$, $w_2=w_3$ for $w_1<\frac{1}{3}w_0$ and $w_1=w_3$ for $w_2<\frac{1}{3}w_0$. Furthermore, the temperature operator in the eight-vertex model corresponds in the AT model to the crossover operator in the $w_0=w_1+w_2+w_3$ plane. So within this plane the free energy becomes singular when the Baxter lines are crossed. This corresponds to phase transitions between the three types of partial de-mixing. In the $w_1\gg w_2$, w_3 region of the plane an AB/CD mixture is obtained. A is mixed with B, and C with D, but there is no mixing between the two groups. In this region the order parameter $\langle s_1\rangle\neq 0$, while the two other natural order parameters of the AT model, $\langle t_i\rangle$ and $\langle s_it_i\rangle$ (Enting 1975a), remain zero (we will denote the conjugated fields by h_s, h_t and h_{st}, respectively). In the $w_2\gg w_1$, w_3 region one finds an AD/BC-mixture with only $\langle t_i\rangle\neq 0$, and in the $w_3\gg w_1$, w_2 region one finds an AD/BC-mixture with only $\langle s_it_i\rangle\neq 0$.

Notice that the permutation symmetry between w_1, w_2 and w_3 (obtained by a permutation of A, B, C and D) is the counterpart in the AT language of equations (2.3*a*) and (2.3*d*). Furthermore, we see that this symmetry maps the three order parameters onto each other. Along $J_1=J_2$ ($w_1=w_2$), Enting (1975a) named (in analogy to the Ising version of the eight-vertex model) $\langle s_it_i\rangle$ the polarisation and $\langle s_i\rangle=\langle t_i\rangle$ the magnetisation.

Along $J_2 = 0$ ($w_2 = w_3$) however, $\langle s_i \rangle$ must be identified as the polarisation and $\langle t_i \rangle = \langle s_i t_i \rangle$ as the magnetisation.

Crossing the dual plane in the temperature direction (varying u), only a singularity in the free energy will be found at the Baxter lines. This is due to the fact that the AT model shows two phase transitions (Wu and Lin 1974). Departing from high temperatures one first finds a transition to one of the three partial de-mixed states (the corresponding order parameter, identified as the polarisation, no longer vanishes). After crossing the dual plane a second transition to total de-mixing takes place (the magnetisations also become nonzero). The two sheets of Ising-type critical points (Wu and Lin 1974, Knops 1975, Aharony 1977, Ashley 1978) are mapped on each other by the duality relation of Fan (1972) (or equivalently by inversion of u). They only intersect each other (in the $w_0 = w_1 + w_2 + w_3$ plane) at the Baxter lines, where the transition need no longer be of the Ising type, but is described by a continuously varying exponent y_T^{AT} (a drawing of the full phase diagram is given by Wu and Lin 1974).

We can conclude that via equations (2.8) the $J_2 = 0$ AT model maps on a staggered F model. The temperature operator of the AT model corresponds to the operator u, i.e. a staggering in the a and b weights. For $a > \frac{1}{2}$ one will find a continuous varying exponent y_T^{AT}, while for $a < \frac{1}{2}$ the free energy will remain regular with respect to u (see table 1). Furthermore the polarisation $\langle s_i \rangle$ is expected to behave in a similar way to the staggered polarisation of the F model: along the Baxter line a continuous varying exponent y_s^{AT} and for $J_1 > 0$ (i.e. $a < \frac{1}{2}$) a first-order transition. The magnetisation $\langle s_i t_i \rangle = \langle t_i \rangle$, on the other hand, remains zero along the whole duality line. One expects along the Baxter line a continuously varying exponent y_H^{AT}, and for $J_1 > 0$ one expects the free energy to remain regular with respect to h_t and h_{st}.

2.3. *The q-state Potts model*

The q-state Potts model, introduced by Potts (1952) can also be considered as a model to describe mixtures: a mixture of q components. On a two-dimensional lattice are situated spins that can take q different states $\sigma_i = 1, 2, 3 \ldots q$. The Hamiltonian is chosen as

$$H = \sum_{\langle ij \rangle} K \delta_{\sigma_i \sigma_j}. \tag{2.10}$$

For $q = 2$ the model reduces to the Ising model, while in the limit $q \to 1$ one obtains the percolation model and in the limit $q \to 0$ (when taken in the appropriate way) a linear resistance network (Fortuin and Kasteleyn 1972). Notice that the AT model for $J_1 = J_2 = 0$ reduces to the four-state Potts model. Further, the three-state Potts model is of interest for adsorption experiments on graphite substrates (Berker *et al* 1978).

In the present form the model only makes sense for integer values of q. In the random cluster model presentation of Fortuin and Kasteleyn (1972), however, q can take all real values. Temperley and Lieb (1971) and Baxter *et al* (1976) showed this model to be equivalent to a staggered F model. The equations read (Baxter 1973b)

$$(\omega_1 \ldots \omega_6) = \begin{cases} (a\,\mathrm{e}^{u},\ a\,\mathrm{e}^{+u},\ a\,\mathrm{e}^{-u},\ a\,\mathrm{e}^{-u},\ \mathrm{e}^{s},\ \mathrm{e}^{-s}) & \text{A sublattice} \\ (a\,\mathrm{e}^{-u},\ a\,\mathrm{e}^{-u},\ a\,\mathrm{e}^{u},\ a\,\mathrm{e}^{u},\ \mathrm{e}^{-s},\ \mathrm{e}^{s}) & \text{B sublattice} \end{cases} \tag{2.10a}$$

with

$$
\begin{aligned}
&a^{-2} = 2(\cosh 2u + \cosh\theta)\\
&2u = \ln x\\
&2s = \ln[(1 + x\,\mathrm{e}^{\theta})/(x + \mathrm{e}^{\theta})]\\
&x = q^{-1/2}(\mathrm{e}^{K} - 1)\\
&\cosh\theta = \tfrac{1}{2}q^{1/2}.
\end{aligned}
\tag{2.10b}
$$

At its critical points $x = 1$ ($K_c(q) = \ln(1+\sqrt{q})$ (Potts 1952)), the Potts model reduces to the F model. Again the critical point of the four-state Potts model maps on the $a = \frac{1}{2}$ point of the F model. All critical points for $q \leq 4$ are mapped onto the Baxter line.

The temperature operator of the Potts model is a combination of the temperature operator of the AT model and the staggered field operator of the F model. Notice that θ is purely imaginary for $q \leq 4$ and real for $q > 4$. From equations (2.10*b*) one finds that u is real for all q. The staggered field s, however, is only real for $q > 4$. For small $\epsilon_P = \ln x$ (i.e. close to the critical point), one finds:

$$
\begin{aligned}
&s \simeq \tfrac{1}{2}\tanh(\theta/2)\epsilon_P\\
&u \simeq \tfrac{1}{2}\epsilon_P.
\end{aligned}
\tag{2.11}
$$

So, close to the Baxter line ($q \leq 4$) s becomes purely imaginary.

For $q > 4$ (the $a < \frac{1}{2}$ domain of the F model), the free energy shows a first-order transition with respect to s and remains regular in the u direction (see above). Because at $x = 1$, ϵ_P is a linear combination of u and s (both real), a first-order transition must also be found in the Potts model for $q > 4$ (Baxter 1973b).

Notice that five different parameters have been used to parametrise the Baxter line: a, Δ, μ, θ and q. They are related by

$$
-\Delta = \cos\mu = \cosh\theta = \tfrac{1}{2}\sqrt{q} = 1/2a^2 - 1. \tag{2.12}
$$

The parameter q can only be applied for $a \leq \frac{1}{2}\sqrt{2}$. At $a = \frac{1}{2}\sqrt{2}$ ($q = 0$) the eight-vertex model and the AT model both reduce to two decoupled Ising models. The staggered F-model is solvable here since the free-fermion condition is satisfied (see e.g. Wu and Lin 1975). Not only the second derivatives of the free energy in the u and d directions diverge logarithmically. The second derivative with respect to the staggered field s also diverges as $\ln|s|$ (Baxter 1970) at this point on the Baxter line. In the Potts direction ϵ_P the free energy behaves as $f \sim m(x)$; i.e. $y_T^{\mathrm{P}} = 0$.

Another special point is the critical point in the four-state Potts model. At least four Baxter lines emerge from this multicritical point: three in the duality plane of the AT model (i.e. the a–b symmetrical eight-vertex model) and one in the q direction of the Potts model (both models are included in the so-called cubic model introduced by Kim *et al* 1976). A similar multicritical point is found in another model (José *et al* 1977). This model can be described as an XY model (in fact a Villain model) with three parameters: the coupling K, a magnetic field h_4 with cubic anisotropy and a parameter y controlling the number of vortices. This model is related to the Coulomb gas (Villain 1975, José *et al* 1977) and to the discrete Gaussian model (Knops 1977).

It is tempting to believe also that this multicritical point is isomorphic to the four-state Potts critical point. The mapping of a modified version of the discrete Gaussian model on the F model given by van Beyeren (1977) gives some support for this idea. Furthermore, Kadanoff (1977) in an heuristic way has conjectured this isomorphism on the basis of a comparison of the expansions of some critical exponents around this multicritical point.

3. Relations between critical exponents along the Baxter line

One of the most intriguing aspects of the solution of the symmetrical eight-vertex model is the breakdown of universality. The critical exponent y_T^{8v} varies continuously along the Baxter line; y_T^{8v} is not independent of the details of the interactions. Kadanoff and Wegner (1971) have shown the existence of a marginal operator along the Baxter line (at least to first-order around $a=\frac{1}{2}\sqrt{2}$, where the eight-vertex and AT model reduce to two decoupled Ising models). In an exact renormalisation transformation one should therefore find (in order to obtain a proper description of the transition), not one fixed point attracting the whole critical domain, but a line of fixed points. Notice that in the eight-vertex and AT language this fixed point line will be difficult to obtain because of the approximations usually needed in the RT equations. In the Potts language, however, q is automatically preserved. Consequently, in the Potts presentation the variation of the critical exponents along the Baxter line is not unexpected. At integer values of q, the transition point has the nature of a q critical point (an endpoint of a line of q coexisting phases).

Along a fixed point line the critical exponents for different operators are in principle allowed to change independently of each other. Along the Baxter line, however, this does not seem to be the case. Simple relations between several exponents are conjectured which are independent of μ. The critical exponents are 'weak universal' (see e.g. the discussion given by Enting 1975b).

For the magnetic and staggered field operator in the eight-vertex model, Barber and Baxter (1973) and Baxter and Kelland (1974) respectively have proposed the relations:

$$y_H^{8v}=\tfrac{15}{8} \tag{3.1}$$

$$y_s^{8v}=\tfrac{3}{2}+\tfrac{1}{4}y_T^{8v} \tag{3.2}$$

(according to scaling: $\beta_i=(2-y_i)/y_T$). Furthermore, Enting (1975b) conjectured similar relations for the AT model:

$$y_H^{AT}=\tfrac{15}{8} \tag{3.3}$$

$$y_s^{AT}=\tfrac{3}{2}+\tfrac{1}{4}y_T^{AT} \tag{3.4}$$

with y_s^{AT} the exponent for the operator conjugated to the polarisation in the AT model. Finally, Kadanoff (1977) has recently proposed the following relation between the temperature exponents of the eight-vertex and AT model

$$(y_T^{8v}-2)(y_T^{AT}-2)=1. \tag{3.5}$$

These relations are known to be correct up to first order (using the method first applied by Kadanoff and Wegner 1971) around $a=\frac{1}{2}\sqrt{2}$, and agree with the exact results for the Ice models. They are conjectured to be exact along the whole Baxter line; this is confirmed by series expansion results. Notice that equations (3.3) and (3.4) are consistent with equation (3.5) at $a=\frac{1}{2}$; they lead in the four-state Potts point to the same y_T^{AT}. The permutation symmetry between the w_i (see § 2) leads in the four-state Potts model to $y_H^{AT}=y_s^{AT}$. So from equations (3.3) and (3.4) Enting (1975a) has conjectured that $y_T^{\rm AT}=\frac{3}{2}$ at the Potts point. From the Baxter solution we know that $y_T^{8\rm v}=0$ at this point. So equation (3.5) also leads to $y_T^{\rm AT}=\frac{3}{2}$ (Kadanoff 1977).

Further it is noteworthy that equations (3.1) and (3.3) are actually the same. One can easily show that the magnetic field operator in the AT model and the eight-vertex model transform into each other. The mapping of the AT model onto a staggered eight-vertex model in equations (2.8) is obtained by a duality transformation on the t_i spins only. The lattice with two Ising spins at every vertex (the AT model) is mapped to an Ising model with s_i spins and t_i spins at the A and B sublattice respectively. The equivalence between the eight-vertex model and the Ising model leads then to equations (2.8).

The s_i spins are unaffected. So the duality transformation can still be applied when extra interactions between the s_i spins only are present. The field h_s leads to an Ising model with a magnetic field on the spins of the A sublattice. In the AT model for $J_1=J_2$, $\langle s_i\rangle$ is identified as the magnetisation. In the corresponding Ising model $\langle s_i\rangle$ is the magnetisation of one sublattice. Along the Baxter line, the Hamiltonian of that Ising model is invariant under permutation of the two sublattices (only four-body coupling and an isotropic next-nearest neighbour interaction). So $\langle s_i\rangle=\langle t_i'\rangle$ and

$$y_H^{\rm AT}=y_H^{8\rm v}. \tag{3.6}$$

Table 2. Comparison between $y_T^{8\rm v}$, $y_T^{\rm AT}$ and $y_T^{\rm P}$ along the Baxter line. q is chosen as the parameter. The values for $y_T^{\rm AT}$ are obtained by equation (3.5) and the values for $y_T^{\rm P}$ are conjectures from Table 3.

q	$y_T^{8\rm v}$	$y_T^{\rm AT}$	$y_T^{\rm P}$
0	1	1	0
1	$\frac{2}{3}$	$\frac{5}{4}$	$\frac{3}{4}$
2	$\frac{1}{2}$	$\frac{4}{3}$	1
3	$\frac{1}{3}$	$\frac{7}{5}$	$\frac{6}{5}$
4	0	$\frac{3}{2}$	$y_T^{\rm AT}$

We now propose a relation, similar to equation (3.5), between $y_T^{8\rm v}$ and $y_T^{\rm P}$. In table 2 the Baxter line is parametrised by q. The exact known values for $y_T^{8\rm v}$ and the values for $y_T^{\rm AT}$ conjectured via equation (3.5) are compared with the $y_T^{\rm P}$ of the Potts model, as obtained by approximate calculations. The rational values for $y_T^{\rm P}$ given in the fourth column can be conjectured from results of recent Monte Carlo calculations, adsorption experiments, series expansions and renormalisation transformations (see table 3). In particular, the Kadanoff lower-bound renormalisation transformation (RT) (Kadanoff 1975, Dasgupta 1976, 1977, den Nijs 1979), suggests these values for $y_T^{\rm P}$. Notice that the values quoted for $q=2$ (Ising) and $q=0$ (spanning trees) are exact results.

From the values in table 2, we can propose the relations

$$y_T^{\rm P}=3(y_T^{\rm AT}-1) \tag{3.7a}$$

$$(y_T^{\rm P}-3)(y_T^{8\rm v}-2)=3. \tag{3.7b}$$

Using Baxter's result ($y_T^{8\rm v}=2\mu/\pi$), an explicit equation for $y_T^{\rm P}(q)$ can be obtained

$$y_T^{\rm P}(q)=\tfrac{3}{2}[2+\pi/(\mu-\pi)] \tag{3.8}$$

Table 3. Values of y_T^P obtained by approximated calculations.

q	$y_T^P = 2/(2-\alpha) = 1/\nu$	Method
1	$0{\cdot}77 \pm 0{\cdot}06$	Monte Carlo (Kirkpatrick 1976)
	0·746	Series expansion (Dunn *et al* 1975)
	0·7496	Series expansion (Domb and Pearse 1976)
	0·738	Renormalisation transform (Reynolds *et al* 1978)
	0·7450	Kadanoff lower-bound RT (Dasgupta 1976)
2	1 (lg)	Exact (Onsager 1944)
	1·0009	Kadanoff lower-bound RT (Kadanoff 1975)
3	1·1696	Series expansion (Zwanzig and Ramshaw 1977)
	1·219	Adsorption experiment (Bretz 1977)
	1·266	Series expansion (de Neef and Enting 1977)
	1·2023	Kadanoff lower-bound RT (Dasgupta 1977, Burkhardt *et al* 1976)

with

$$\cos\mu = \tfrac{1}{2}\sqrt{q}.$$

This function is drawn in figure 3. Notice that at $q = 4$ the derivative $dy_T^P/dq \to \infty$. For small q equation (3.8) can be approximated by $y_T^P = 3/\pi\sqrt{q}$. This agrees with the result of Kunz and van Leeuwen (Kunz 1977). Using a Migdal approximation for small q, they found $y_T^P = \sqrt{q}$. Equation (3.8) however disagrees with the recent conjecture of Klein *et al* (1978) for the critical exponent of the percolation model $y_T^P(1) = 2 \ln \frac{3}{2}/\ln 3 = 0{\cdot}738$.

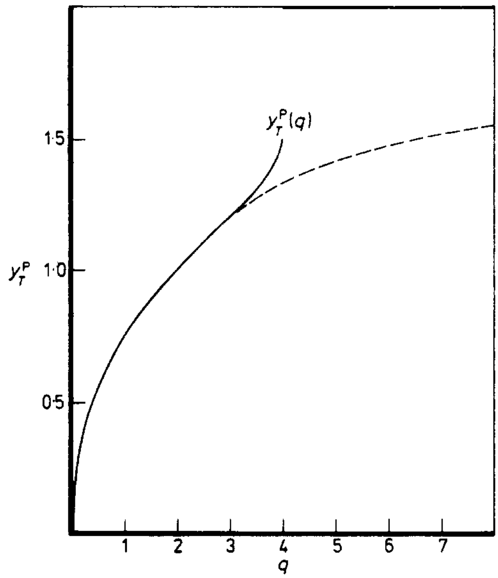

Figure 3. The critical exponent y_T^P as function of q. The full curve is obtained from equation (3.8) and the broken curve is obtained by the Kadanoff lower-bound method (for $q < 3$ the broken curve is not drawn because it practically coincides with the full curve).

The $y_T^P(q)$ obtained by the Kadanoff lowerbound RT shows an excellent correspondence with equation (3.8) up to $q=3$. For $q>3$, however, the results of the RT disagree with equation (3.8) (see the broken curve in figure 3. For $q<3$ the broken curve is not shown because it almost coincides with the drawn curve. The drawn curve shows a point of inflection at $q=2{\cdot}7318$). One of the features of all RTs constructed up to now for the Potts model (Aharony 1977, Dasgupta 1977, den Nijs 1979, Shenker *et al* 1979) is that they are unable to describe the first-order transition for $q>4$. The transition is found to remain critical. The $y_T^P(q)$ curve remains smooth around $q=4$ (see figure 3) and predicts $y_T^P(4)=\frac{4}{3}$. This result of the RT at $q=4$ based on a straightforward application of the variational method is questionable. Arguments can be given for another solution yielding $y_T^P(4)\simeq\frac{3}{2}$ (den Nijs 1979).

In conclusion, we have seen in this paper that the critical line in the four-state Potts model (up to $q=4$) can be mapped onto the Baxter line. Furthermore an explicit equation for the critical exponent $y_T^P(q)$ is proposed. Because this equation confirms weak universality and also gives a good fit for the y_T^P obtained by approximate calculations, equation (3.8) is expected to be exact.

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